Engineering Maths First Aid Kit

5.3

Multiplying matrices

Introduction

One of the most important operations carried out with matrices is **matrix multiplication**. At first sight this is done in a rather strange way. The reason for this only becomes apparent when matrices are used to solve equations.

1. Some simple examples

To multiply $\begin{pmatrix} 3 & 7 \end{pmatrix}$ by $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$ perform the following calculation.

$$\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = 3 \times 2 \quad + \quad 7 \times 9 = 6 + 63 = 69$$

Note that we have paired elements in the row of the first matrix with elements in the column of the second matrix, multiplied the paired elements together and added the results.

Another, larger example:

$$\begin{pmatrix} 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix} = 4 \times 3 \quad + \quad 2 \times 6 \quad + \quad 5 \times 8 = 12 + 12 + 40 = 64$$

Exercises

1. Evaluate the following:

a) $\begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$, b) $\begin{pmatrix} -3 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$, d) $\begin{pmatrix} -4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ -8 \end{pmatrix}$.

Answers

1. a) 53, b) 57, c) 64, d) -40.

2. More general matrix multiplication

When we multiplied matrices in the previous section the answers were always single numbers. Usually however, the result of multiplying two matrices is another matrix. Two matrices can only be multiplied together if the number of columns in the first matrix is the same as the number of rows in the second. So, if the first matrix has size $p \times q$, that is, it has p rows and q columns, and the second has size $r \times s$, that is, it has r rows and s columns, we can only multiply them together if $q = r$. When this is so, the result of multiplying them together is a $p \times s$ matrix.

Two matrices can only ever be multiplied together if the number of columns in the first is the same as the number of rows in the second.

Example

Find $\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix}$.

Solution

The first matrix has size 2×2. The second has size 2×1. Clearly the number of columns in the first is the same as the number of rows in the second. So, multiplication is possible and the result will be a 2×1 matrix. The calculation is performed using the same operations as in the examples in the previous section.

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

To obtain the first entry in the solution, ignore the second row of the first matrix. You have already seen the required calculations.

$$\begin{pmatrix} 3 & 7 \\ & \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ \end{pmatrix}$$

To obtain the second entry in the solution, ignore the first row of the first matrix.

$$\begin{pmatrix} & \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} \\ 53 \end{pmatrix}$$

Putting it all together

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 69 \\ 53 \end{pmatrix}$$

Example

Find $\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix}$.

Solution

The first matrix has size 2×2. The second matrix has size 2×2. Clearly the number of columns in the first is the same as the number of rows in the second. The multiplication can be performed and the result will be a 2×2 matrix.

$$\begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 4 \times (-1) & 2 \times 6 + 4 \times 9 \\ 5 \times 3 + 3 \times (-1) & 5 \times 6 + 3 \times 9 \end{pmatrix} = \begin{pmatrix} 2 & 48 \\ 12 & 57 \end{pmatrix}$$

Exercises

1. Evaluate the following.

a) $\begin{pmatrix} -3 & 2 \\ 3 & 11 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 2 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ -1 & 9 \end{pmatrix}$, c) $\begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 & 2 \\ 5 & 13 & 1 \end{pmatrix}$.

Answers

1. a) $\begin{pmatrix} -11 \\ -2 \end{pmatrix}$, b) $\begin{pmatrix} 10 & 58 \\ 4 & 149 \end{pmatrix}$, c) $\begin{pmatrix} 9 & 13 & 5 \\ 47 & 117 & 11 \end{pmatrix}$.

